ROBOTICS:
JACOBIANS: VELOCITIES AND STATIC FORCES

Sami Haddadin

Institute of Robotics and Mechatronics
German Aerospace Center (DLR), Germany

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Overview

Up to now: static problems

1. representation of orientation
2. forward kinematics
3. inverse kinematics

Now: velocities

1. angular and linear velocity of a rigid body
2. motion of a manipulator
3. forces acting on a robot
4. **Jacobian**

Notation I

derivative of a vector:

$$ {}^{B}V_{Q} := \frac{d}{\mathrm{d}t} {}^{B}Q = \lim_{\Delta t \to 0} \frac{{}^{B}Q(t+\Delta t) -^{B} Q(t)}{\Delta t} \tag{1} $$

It is important to indicate the frame in which the vector is differentiated.

vector expressed in terms of frame $\{A\}$:

$$ {}^{A}({}^{B}V_{Q}) = \frac{{}^{A}d}{\mathrm{d}t} {}^{B}Q \tag{2} $$

$\rightarrow$ velocity vector is associated with a point in space, but the numerical values of the velocity depend on two frames:

1. one to which the differentiation was done
2. one in which the resulting velocity vector is expressed

Notation II

remove the outer, leading superscript by explicitly including the rotation matrix ${}^{A}R_{B}$:

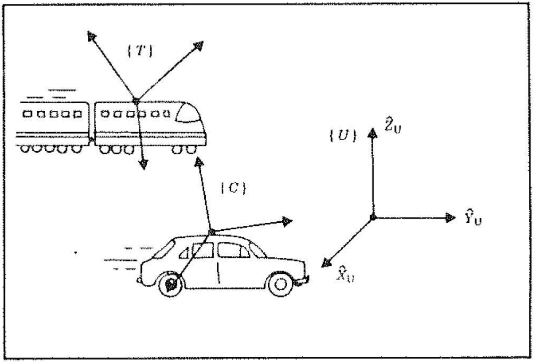
$$ {}^{A}({}^{B}V_{Q}) = {}^{A}R_{B}\,{}^{B}V_{Q} \tag{3} $$

We use mostly the right hand side expression!
Important special case: **velocity of the origin of a frame** relative to some understood universe reference frame $\{U\}$.

$$ v_{C} = {}^{U}V_{C,org}, \tag{4} $$

where the point in question is the origin of $\{C\}$. ${}^{A}v_{C}$ is the velocity of the origin of $\{C\}$ expressed in terms of $\{A\}$. Note that the differentiation is done relative to $\{U\}$.

Frames in linear motion



Example I

1. universe frame $\{U\}$
2. frame $\{T\}$ attached to train traveling at 100 km/h
3. frame $\{C\}$ attached to car traveling at 30 km/h
4. both vehicles head in $\hat{X}$ direction of $\{U\}$
5. ${}^{U}R_{T}$ and ${}^{U}R_{C}$ are known

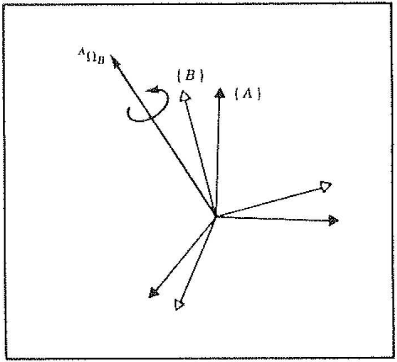
Example II

What is $\frac{{}^U d}{\mathrm{d}t} {}^U P_{C,org}$?

What is ${}^C({}^U V_{T,org})$?

What is ${}^C({}^T V_{C,org})$?

Relative angular motion



The angular velocity vector I

Whereas linear velocity describes an attribute of a point, angular velocity describes an attribute of a body. Since we always attach a frame to the bodies we consider, we can also think of angular velocity as describing the rotational motion of a frame.

- ${}^{A}\Omega_{B}$ describes the rotation of $\{B\}$ relative to $\{A\}$
- at any instant, the direction of ${}^{A}\Omega_{B}$ indicates the instantaneous axis of rotation of $\{B\}$ relative to $\{A\}$
- magnitude of ${}^{A}\Omega_{B}$ indicates the speed of rotation
- like any other vector, an angular velocity vector may be expressed in any coordinate system $\{C\}$: ${}^{C}({}^{A}\Omega_{B})$

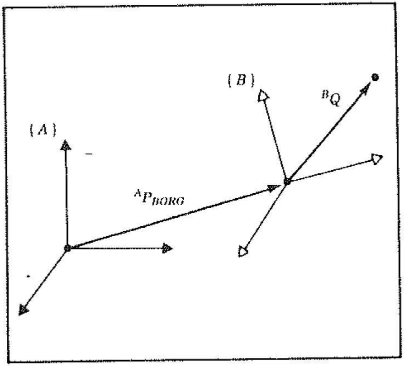
The angular velocity vector II

Introduce a simplified notation for the important special cased of a well understood reference frame:

$$\omega_C = {}^U\Omega_C \tag{5}$$

again: ${}^A\omega_C$ is the angular velocity of $\{C\}$ expressed in terms of $\{A\}$, though the angular velocity is with respect to $\{U\}$.

Relative translation



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Linear velocity of rigid bodies

Consider frame $\{B\}$ attached to RB. We wish to describe the motion of ${}^{B}Q$ relative to frame $\{A\}$. Consider $\{A\}$ to be fixed.

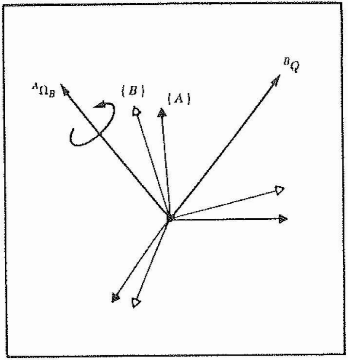
- Frame $\{A\}$ is located relative to $\{A\}$ described by ${}^{A}P_{B,org}$ and ${}^{A}R_{B}$
- **for the moment:** ${}^{A}R_{B} \neq {}^{A}R_{B}(t)$
- $\rightarrow$ motion of Q relative to $\{A\}$ is due to ${}^{A}P_{B,org}$ and/or ${}^{B}Q$ changing in time

Solving for the linear velocity of Q in terms of $\{A\}$:

$$ {}^{A}V_{Q} = {}^{A}V_{B,org} + {}^{A}R_{B}\,{}^{B}V_{Q} \tag{6} $$

only valid for the case in which relative orientation of $\{B\}$ and $\{A\}$ remains constant

Relative rotation of vector



Rotational velocity of rigid bodies I

Consider two frames with coincident origins and with zero linear velocity so the origins will remain coincident for all time.

- orientation of $\{B\}$ with respect to $\{A\}$ is changing in time
- rotational velocity of $\{B\}$ relative to $\{A\}$ is described by ${}^{A}\Omega_{B}$
- ${}^{B}Q$ locates a point which is fixed in $\{B\}$

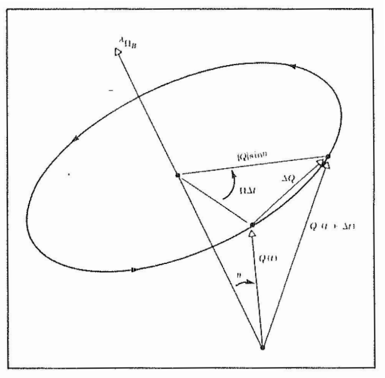
Question: How does a vector change with time as viewed from $\{A\}$ when it is fixed in $\{B\}$ and the systems are rotating?

Rotational velocity of rigid bodies II

- consider ${}^{B}V = 0$
- **still:** point Q will have a velocity as seen from $\{A\}$ due to ${}^{A}\Omega_B$

For now: intuitive approach

Velocity of a point



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Rotational velocity of rigid bodies III

- two instants of time as vector Q rotates around ${}^{A}\Omega_{B}$ (exactly what observer $\{A\}$ would see)
- it is clear that the differential change in ${}^{A}Q$ must be perpendicular to both ${}^{A}\Omega_{B}$ and ${}^{A}Q$

From observation we see that

$$|\Delta Q| = (|{}^{A}Q| sin(\theta))(|{}^{A}\Omega_{B}|\Delta t) \tag{7}$$

$\rightarrow$ these observations from magnitude and direction immediately suggest the vector cross product:

$${}^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}Q \tag{8}$$

Rotational velocity of rigid bodies IV

In general Q may also be changing with respect to $\{B\}$:

$$^{A}V_{Q} = {}^{A}({}^{B}V_{Q}) + {}^{A}\Omega_{B} \times {}^{A}Q \tag{9}$$

Using a rotation matrix to remove the dual-superscript together with ${}^{A}Q = {}^{A}R_{B}{}^{B}Q$ we get

$$^{A}V_{Q} = {}^{A}R_{B}{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}{}^{B}Q \tag{10}$$

Simultaneous linear and angular velocity

for the case where origins are not coincident we simply add the linear velocity of the origin

$$ {}^{A}V_{Q} = {}^{A}V_{B,org} + {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q \qquad (11) $$

$\rightarrow$ general form for a velocity of a vector fixed in frame $\{B\}$ as seen from frame $\{A\}$

Properties of angular velocity I

- **up to now:** geometric approach
- **now:** mathematical approach

property of $\dot{R}$:

$$
\begin{aligned}
RR^T &= I_n & (12)\\
\dot{R}R^T + R\dot{R}^T &= 0_n & (13)\\
\dot{R}R^T + (\dot{R}R^T)^T &= 0_n, & (14)
\end{aligned}
$$

with I_n being the $n \times n$ identity matrix and 0_n being the $n \times n$ zero matrix.

Definition:

$$S := \dot{R}R^T \tag{15}$$

Now (14) can be written as

$$S + S^T = 0_n. \tag{16}$$

$\rightarrow$ S is skew-symmetric (skew symmetry: $A = -A^T$).

Properties of angular velocity II

Now we have a property relating the derivative of orthonormal matrices with skew-symmetric matrices (remember: $R^{-1} = R^T$):

$$S = \dot{R} R^{-1} \tag{17}$$

Properties of angular velocity III

How can we use this result? $\rightarrow$ velocity of a point due to rotating reference frame

Consider a fixed vector ${}^{B}P$ with respect to frame $\{B\}$. Its description in another frame $\{A\}$ is given as

$$ {}^{A}P = {}^{A}R_{B}\,{}^{B}P, \tag{18} $$

If frame $\{B\}$ is rotating (i.e. ${}^{A}\dot{R}_{B} \neq 0$) then ${}^{B}P$ will be changing even though ${}^{B}P$ is constant.

$$ {}^{A}\dot{P} = {}^{A}\dot{R}_{B}\,{}^{B}P, \tag{19} $$

or using our notion of velocity

$$ {}^{A}V_{P} = {}^{A}\dot{R}_{B}\,{}^{B}P. \tag{20} $$

Properties of angular velocity IV

$$ {}^{A}V_{P} = {}^{A}\dot{R}_{B}\, {}^{A}R_{B}^{-1}\, {}^{A}P. \tag{21} $$

Making use of (17) we obtain

$$ {}^{A}V_{P} = {}^{A}S_{B}\, {}^{A}P. \tag{22} $$

The subscripts of S are used for indicating the relation to ${}^{A}R_{B}$.
!!The matrix S is called the **angular velocity matrix**!!

Properties of angular velocity V

Skew-symmetric matrices and the vector cross product Assign the elements in a skew symmetric matrix S as follows:

$$S := \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \tag{23}$$

and define the 3×1 column vector

$$\Omega := \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}, \tag{24}$$

then it is easily verified that

$$SP = \Omega P, \tag{25}$$

where P is a vector, and $\times$ is the vector cross product.

Properties of angular velocity V

Ω is called the **angular velocity vector** (see previous slides). This makes it possible to write (22) as

$$ {}^{A}V_{P} = {}^{A}\Omega_{B} \times {}^{A}P. \tag{26} $$

$\rightarrow$ it is the angular velocity vector which specifies the motion of $\{B\}$ with respect to $\{A\}$.

Properties of angular velocity VI

question: What is the physical meaning of Ω?
$\rightarrow$ we derive Ω by direct differentiation of R:

$$\dot{R} = \lim_{\Delta t \rightarrow 0} \frac{R(t+\Delta t) - R(t)}{\Delta t} \tag{27}$$

$R(t+\Delta t)$ is now written as the concatenation of two matrices:

$$R(t+\Delta t) = R_{\hat{K}}(\Delta\theta)R(t), \tag{28}$$

where over the interval Δt a small rotation of $\Delta\theta$ has occurred about $\hat{K}$. (27) and (28) lead to

$$\dot{R} = \lim_{\Delta t \rightarrow 0} \left(\frac{R_{\hat{K}}(\Delta\theta) - I_3}{\Delta t} R(t) \right) \tag{29}$$

$$\dot{R} = \lim_{\Delta t \rightarrow 0} \left(\frac{R_{\hat{K}}(\Delta\theta) - I_3}{\Delta t} \right) R(t) \tag{30}$$

Properties of angular velocity VII

with small angle substitution and usage of the "angle-axis" rotation matrix

$$R_{\hat{K}}(\theta) = \begin{bmatrix} k_x^2 v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y^2 v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z^2 v\theta + c\theta \end{bmatrix}, \quad (31)$$

with $v\theta = -cos(\theta)$