3.2.3 Hybrid Equations

Once a *state vector* $\boldsymbol{x}$ is chosen, a complete model of the *state functions* of both, the system and the measurement process has to be considered, in order to obtain an appropriate formulation of the EKF. To mirror the discrete character of the motion – which leads to an alternating reference frame in the feet – the filter method requires an adaptation. *Hybrid* System State Equations – both, for step performance period and for standing foot change instants – can be easily defined for the *state vector* including the robot model errors defined in Section 2.1. This allows a correct linearization of the error components so that the complete system evolution can be estimated.

For $S_{F_{R,L}}$ fixed on the ground, the *state transition function* $\boldsymbol{f}$ can be represented as follows:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} {}_0\boldsymbol{x}_{0_{k+1}} \\ {}_F\boldsymbol{x}_{1_{k+1}} \\ \vdots \end{bmatrix} = \boldsymbol{f}\left(\boldsymbol{x}_k, \boldsymbol{u}_{k+1}, \boldsymbol{w}_k\right) = \boldsymbol{x}_k\left(1-\gamma_{k+1}\right) + \boldsymbol{f}_s\left(\boldsymbol{x}_k, \boldsymbol{w}_k\right)\gamma_{k+1} \tag{3.17}$$

where γ_{k+1} is a binary variable representing the current control vector: $\gamma_{k+1} = 0$ means no change in the coordinate frame, $\gamma_{k+1} = 1$ means change in the coordinate frame. Consequently, the function $\boldsymbol{f}_s$ is defined as the transformation of the *state vector* $\boldsymbol{x}$ when a step is performed. Equation (3.17) reflects the *hybrid nature* of the approach.

According to the considerations exposed in Chapter 2 and the ones expressed in this chapter, $\boldsymbol{f}_s$ is modeled the following way:

$$\boldsymbol{f}_s(\boldsymbol{x}_k, \boldsymbol{w}_k) = \begin{bmatrix} \begin{pmatrix} \begin{bmatrix} x_{0_k} \\ y_{0_k} \end{bmatrix} + \boldsymbol{Rot}_{2\times 2}\left(\theta_k\right) \begin{bmatrix} x_{s_k} + \Delta x_{s_k} \\ y_{s_k} + \Delta y_{s_k} \end{bmatrix} \\ \theta_k + \theta_{s_k} + \Delta\theta_{s_k} \end{pmatrix}_{3\times 1} \\ \\ \boldsymbol{Rot}_{3\times 3}\left(\boldsymbol{z}_F, -(\theta_{s_k}+\Delta\theta_{s_k})\right) \left(\begin{bmatrix} x_{1_k} \\ y_{1_k} \\ z_{1_k} \end{bmatrix} - \begin{bmatrix} x_{s_k} + \Delta x_{s_k} \\ y_{s_k} + \Delta y_{s_k} \\ 0 \end{bmatrix} \right)_{3\times 1} \\ \\ \vdots \end{bmatrix} \tag{3.18}$$

where x_s and y_s the coordinates of the step length in the previous foot coordinate frame, θ_s the desired orientation change in the step, $\boldsymbol{Rot}_{3\times3}(\boldsymbol{z},\theta)$ the three-dimensional rotation matrix around vector $\boldsymbol{z}$ of θ radians, $\boldsymbol{Rot}_{2\times2}(\theta)$ the bidimensional rotation matrix around perpendicular vector $\boldsymbol{z}_F$ of θ radians, and $\boldsymbol{w} = [\Delta x_s \, \Delta y_s \, \Delta\theta_s]^T$ the robot *dead-reckoning* error sources as explained in Section 2.1.2.

On the other hand, an appropriate measurement equation must be chosen based both on the stereo-vision system properties and the object features detected by it. The calculation of the stereo-vision error covariance matrices has already been explained in the context of (2.7). Measurement equation is modeled the following way:

$$
\begin{aligned}
\boldsymbol{z}_k &= \begin{bmatrix} {}_F\boldsymbol{z}_{L_{1k}} \\ \vdots \\ {}_F\boldsymbol{z}_{O_{1k}} \\ \vdots \end{bmatrix} = \boldsymbol{h}(\boldsymbol{x}_k, \boldsymbol{v}_k) = \\
&= \begin{bmatrix} \boldsymbol{Rot}_{2\times2}\left(-\theta_k\right) \left(\begin{bmatrix} L_{1x} \\ L_{1y} \end{bmatrix} - \begin{bmatrix} x_{0_k} \\ y_{0_k} \end{bmatrix} \right)_{2\times1} \\ \vdots \\ {}_F\boldsymbol{x}_{1_k\,3\times1} \\ \vdots \end{bmatrix} + {}_p\boldsymbol{v}_k
\end{aligned} \tag{3.19}
$$

where ${}_F\boldsymbol{z}_{L_i} = [x_{L_i} \, y_{L_i}]^T$ and ${}_F\boldsymbol{z}_{O_i} = [x_{O_i} \, y_{O_i} \, z_{O_i}]^T$ are the measurements of landmarks and objects respectively and ${}_0\boldsymbol{L}_j = [L_{j_x} \, L_{j_y}]$ the position in world coordinate frame S_0 of the landmarks.

The measurement equation (3.19) has been established using directly the relative positions of the landmarks and objects in relation to the foot frame S_F, as the appropriate measurement transformations have already been made in Chapter 2. ${}_p\boldsymbol{v}$ is the noise exclusively of the perception system – for the EKF only its covariance (Section 2.2.3) must be considered.

The values of the noises in the equations (3.18) and (3.19) being unknown, we can only consider *a priori estimations*, as in (3.7) and (3.8). *Monte Carlo* simulations questioning the validation of the performed linearizations in Section 3.4.4.

Also, the covariance matrix will have to be estimated *a priori* according to (3.9). It also has two different formulations depending on the value of γ_{k+1}, i.e. :

$$\begin{aligned} \boldsymbol{C}_{k+1|k} &= \boldsymbol{C}_{k|k}, & \text{if } \gamma_{k+1} = 0 \qquad (3.20) \\ \boldsymbol{C}_{k+1|k} &= \boldsymbol{A}_{k|\,step}\, \boldsymbol{C}_{k|k}\, \boldsymbol{A}^T_{k|\,step} + \boldsymbol{W}_{k|\,step}\, \boldsymbol{Q}_k\, \boldsymbol{W}^T_{k|\,step}, & \text{if } \gamma_{k+1} = 1 \qquad (3.21) \end{aligned}$$

where, in case of step

$$\boldsymbol{A}_{k|\,step} = \left.\frac{\partial \boldsymbol{f}_s}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{w}\to\boldsymbol{0}} = \begin{bmatrix} 1 & 0 & -(x_{s_k}\, s\,\theta_k + y_{s_k}\, c\,\theta_k) & & \\ 0 & 1 & (x_{s_k}\, c\,\theta_k - y_{s_k}\, s\,\theta_k) & \boldsymbol{0}_{3\times3} & \cdots \\ 0 & 0 & 1 & & \\ & \boldsymbol{0}_{3\times3} & & \boldsymbol{Rot}_{3\times3}\,(\boldsymbol{z}_F, -\theta_{s_k}) & \\ & \vdots & & & \ddots \end{bmatrix} \tag{3.22}$$

$$\boldsymbol{W}_{k|\,step} = \left.\frac{\partial \boldsymbol{f}_s}{\partial \boldsymbol{w}_k}\right|_{\boldsymbol{w}\to\boldsymbol{0}} = \begin{bmatrix} c\,\theta_k & -s\,\theta_k & 0 \\ s\,\theta_k & c\,\theta_k & 0 \\ 0 & 0 & 1 \\ -c(-\theta_{s_k}) & s(-\theta_{s_k}) & s(-\theta_{s_k})(x_{1_k} - x_{s_k}) + c(-\theta_{s_k})(y_{1_k} - y_{s_k}) \\ -s(-\theta_{s_k}) & -c(-\theta_{s_k}) & -c(-\theta_{s_k})(x_{1_k} - x_{s_k}) + s(-\theta_{s_k})(y_{1_k} - y_{s_k}) \\ 0 & 0 & 0 \\ & \vdots & \end{bmatrix} \tag{3.23}$$

$$\boldsymbol{Q}_k = \begin{bmatrix} \sigma^2_{\Delta x_s} & 0 & 0 \\ 0 & \sigma^2_{\Delta y_s} & 0 \\ 0 & 0 & \sigma^2_{\Delta \theta_s} \end{bmatrix} \tag{3.24}$$

where $s\,\theta_k = sin\,\theta_k$, $c\,\theta_k = cos\,\theta_k$, $c(-\theta_{s_k}) = cos\,(-\theta_{s_k})$, and the process noise $\boldsymbol{Q}_k$ represents the *dead-reckoning* error variance accumulated during a single step – see Section 2.1.2. The major advantage of the novel Filter formulation is, that $\boldsymbol{W}_{k|\,step}\,\boldsymbol{Q}_k\,\boldsymbol{W}^T_{k|\,step}$ is included at once, when the current step terminates and the foot frame S_F changes.

For clarify, notice what the EKF does to the component x_0 of the robot's position. Developing (3.21) (without consideration of noises and changes of orientation to make it simple) we obtain for its first value

$$\begin{aligned} C_{x_0 x_0\, k+1|k} &= C_{x_0 x_0\, k|k} + (-x_{s_k}\, sin\, \theta_{k|k} - y_{s_k}\, cos\, \theta_{k|k})^2\, C_{\theta\theta\, k|k} + \\ &\quad + 2\,(-x_{s_k}\, sin\, \theta_{k|k} - y_{s_k}\, cos\, \theta_{k|k})\, C_{x_0\theta\, k|k} \end{aligned} \tag{3.25}$$

which, as shown in (3.14), corresponds to a correct statistical calculation of

$$C\left(x_{0\, k|k} - (\, x_{s_k}\, sin\, \theta_{k|k} + y_{s_k}\, cos\, \theta_{k|k}\,)\, \theta_{k|k} \right) \tag{3.26}$$

the variance in x_0 after a step, linearizing in θ as can be seen in (3.18). The same verifications may be done to calculate the other parameters of the *Covariance Matrix.*

Then the *Covariance Innovation* and the *Kalman Gain* will be calculated according to (3.12) and (3.13), where[3]:

$$\boldsymbol{H}_{k|\,step} = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{v}\to\boldsymbol{0}} = \begin{bmatrix} \begin{matrix} -c\,\theta_k & -s\,\theta_k & -(L_{1x} - x_{0_k})\, s\,\theta_k + (L_{1y} - y_{0_k})\, c\,\theta_k \\ s\,\theta_k & -c\,\theta_k & -(L_{1x} - x_{0_k})\, c\,\theta_k - (L_{1y} - y_{0_k})\, s\,\theta_k \end{matrix} & \boldsymbol{0}_{2\times 3} & \cdots \\ \vdots & \vdots & \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{I}_{3\times 3} & \\ \vdots & & \ddots \end{bmatrix} \tag{3.27}$$

$$\boldsymbol{V}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{V}^T_{k+1} = \begin{bmatrix} \boldsymbol{M}_{L_1\, k+1} & \cdots & \boldsymbol{0} & \cdots \\ \vdots & \ddots & \vdots & \\ \boldsymbol{0} & \cdots & \boldsymbol{M}_{O_1\, k+1} & \\ \vdots & & & \ddots \end{bmatrix} \tag{3.28}$$

where $\boldsymbol{M}_{L_i}$ is bidimensional and $\boldsymbol{M}_{O_i}$ three-dimensional. The measurement covariance sub-matrices $\boldsymbol{M}$ are calculated as explained in Section 2.2.3 – Equation (2.7), starting from the errors in the image frame S_I and afterwards transformed into the foot frame S_F. These matrices cannot be considered as constant as they depend on the object position. Hence, they have to be recalculated at every time step. The relationship between the measurement *not-homogeneous* pixel error $\boldsymbol{v}$ and the x-covariance is neglected by means of the 1st order linearization. To finally substitute in the final expressions (3.10) and (3.11).

[3] Notice the possible extension by the introduction of cross-covariances between measurements in the noise term $\boldsymbol{V}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{V}^T_{k+1}$ for measurements at the same time – see Section 4.1.2.