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Virtual Physics

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Exercise 11: Integration Methods

Task 1: (from Exam WS 2010/2011)

Below you find the Butcher Tableau of an RK method of 3rd order.

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 \\
2/3 & 0 & 2/3 & 0 \\
\hline
1 & 1/4 & 0 & 3/4
\end{array}
$$

Perform one integration step of this method on the following system!

$dx/dt \quad = -x^2 - 2 + 3t$

Start at t=0 with $x_{t=0} = 1$. The step-size h is 1.

Return the result for $x_{t=1}$ as well as for the two-substeps

Compute with rational numbers.

Solution:

$$\dot{x}_{t=0} = -x_{t=0}^2 - 2 + 3t = -1 - 2 + 0 = -3$$

$$x_{P1} = x_{t=0} + \frac{h}{3}\dot{x}_{t=0} = 1 - \frac{1}{3}3 = 0$$

$$\dot{x}_{P1|t=\frac{1}{3}} = -x_{P1}^2 - 2 + 3t = 0 - 2 + 1 = -1$$

$$x_{P2} = x_{t=0} + \frac{2h}{3}\dot{x}_{P1} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\dot{x}_{P2|t=\frac{2}{3}} = -x_{P2}^2 - 2 + 3t = -\frac{1}{9} - 2 + 2 = -\frac{1}{9}$$

$$x_{t=1} = x_{t=0} + \frac{h}{4}\dot{x}_{t=0} + \frac{3h}{4}\dot{x}_{P2} = 1 - \frac{3}{4} - \frac{3}{4\cdot 9} = \frac{12}{12} - \frac{9}{12} - \frac{1}{12} = \frac{1}{6}$$

Task 2: (from Exam WS 2011/2012)

Below you find the coefficients for the BDF methods of different orders.

	α_{t+h}	α_t	α_{t-h}	α_{t-2h}	α_{t-3h}
BDF 1	1	-1			
BDF 2	3/2	-2	1/2		
BDF 3	11/6	-3	3/2	-1/3	
BDF 4	25/12	-4	3	-4/3	1/4

Perform 3 integration steps of the highest applicable BDF method on the following system!

$dx/dt = 2x - t + 1$

Start at t=0 with $x_{t=0} = -1$. The step-size h is 1.

Return the result for $x_{t=1}$, $x_{t=2}$, and $x_{t=3}$.

Compute with rational numbers.

Solution: We use short-hand notation x_a for $x_{t=a}$

Step 1 with BDF1

$$x_1 - x_0 = h(2x_1 - t + 1) = 2x_1$$

$$x_1 = -x_0 = 1$$

Step 2 with BDF2

$$\frac{3}{2}x_2 - 2x_1 + \frac{1}{2}x_0 = h(2x_2 - t + 1)$$

$$\frac{3}{2}x_2 - 2 - \frac{1}{2} = 2x_2 - 2 + 1$$

$$-\frac{1}{2}x_2 = \frac{3}{2}$$

$$x_2 = -3$$

Step 3 with BDF3

$$\frac{11}{6}x_3 - 3x_2 + \frac{3}{2}x_1 - \frac{1}{3}x_0 = h(2x_3 - t + 1)$$

$$\frac{11}{6}x_3 + 9 + \frac{3}{2} + \frac{1}{3} = 2x_3 - 2$$

$$-\frac{1}{6}x_3 = -11 - \frac{3}{2} - \frac{1}{3}$$

$$x_3 = 66 + 9 + 2 = 77$$